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HIGH CONNECTIVITY KEEPING SETS IN n-CONNECTED GRAPHS

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It is proved that for every positive integer k, every n-connected graph G of sufficiently large order contains a set W of k vertices such that $G-W$ is $(n-2)$-connected. It is shown that this does not remain true if we add the condition that $G(W)$ is connected.

Introduction

A graph G is called (n,k)-*critical* or an (n,k)-*graph*, if $\kappa(G-W)=n-|W|$ for all vertex sets W of G with $|W|\le k$, where $\kappa(H)$ denotes the connectivity number of a graph H and n,k are positive integers. It was proved in [13] that every $(n,2)$-graph is finite. This means that an infinite, n-connected graph G contains a set $\{x,y\}$ of vertices such that $G-\{x,y\}$ is $(n-1)$-connected. It was proved in [16] that an infinite, n-connected graph G even contains an infinite set of vertices S such that $G-S'$ is $(n-1)$-connected for all $S'\subseteq S$. This was generalized further in [20] and [4].

It was also proved in [13] that for every n, there is only a finite number of (non-isomorphic) $(n,3)$-graphs. This means that every n-connected graph G of sufficiently large order contains three vertices x,y,z so that $G-\{x,y,z\}$ is $(n-2)$-connected. We will generalize this in section 2 of the present paper in the following way: *For every positive integer k, every n-connected graph G of sufficiently large order contains a set W of k vertices such that $G-W$ is $(n-2)$-connected.* It is shown by examples that for $n\ge 4$, we cannot replace

Mathematics Subject Classification (2000): 05C40, 05C35 in this result $n-2$ with $n-1$, and it is proved in section 3 that, in general, we cannot replace W with a connected subgraph W.

One should mention some results with the same tendency. It was conjectured in [17] that for every positive integer k, every 3-connected graph G of sufficiently large finite order contains a connected subgraph W on k vertices such that $G-V(W)$ is 2-connected. This conjecture was known to be true for $k=2$ from [21] and proved for $k=3$ in [17] and $k=4$ in [7]. Without the condition of connectivity for W, it was proved for all k in [8]. – It was proved implicitly (claimed by T. Böhme) in [1] that for each positive integer k, every maximal planar, 5-connected, finite graph G of sufficiently large order contains a $W \subseteq V(G)$ with $|W|=k$ such that $G-W$ is 4-connected.

We will attach here some general notation, more specific one is introduced in the next section. A *graph* $G=(V,E)$ in this paper is always *undirected without multiple edges and loops.* Let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively, and let $|G| := |V(G)|$ and $||G|| := |E(G)|$. The edge between vertices x and y is denoted by $[x,y]=[y,x]$, and for an $F \subseteq V(G)$ or a subgraph $F \subseteq G, E_G(F) := \{[x,y] \in E(G) : x \in F \text{ and } y \notin F\}$, where $x \in H$ always means $x \in V(H)$ for a graph H. Whereas, for $F,H \subseteq G$, we define $F \cap H := (V(F) \cap V(H), E(F) \cap E(H)) \subseteq G$, we write $F \cap H = \emptyset$ instead of $V(F) \cap V(H) = \emptyset$ and $F \cap S := V(F) \cap S$ for $F \subseteq G$ and $S \subseteq V(G)$. For $F \subseteq G, S := V(F)$, and $E' \subseteq E(G)$, let $F - E' := (V(F), E(F) - E'), G(F) := G(S)$ be the subgraph induced by S in G, and $G - F := G - S := G(V(G) - S)$. For $F \subseteq G$ or $F \subseteq V(G), N_G(F) := \{x \in G - F :$ *there is an* $[x,y] \in E(G)$ *with* $y \in F\}$, and for $x \in G, N_G(x) := N_G(\{x\})$ (corresponding simplification in the other notation) and $\overline{N}_G(x) := N_G(x) \cup \{x\}$. Furthermore, $d_G(x) := |N_G(x)|$ is the degree of the vertex x in $G, \delta(G)$ is the minimum degree of G, and let $V_k(G) := \{x \in G : d_G(x) = k\}$. In the notation $N_G(X), d_G(x)$ etc. we suppress the subcript, if the graph considered is clear from the context. The set of components of the graph G is denoted by $\mathcal{C}(G)$. Let $\mathbb{N}, \mathbb{N}_k, \mathbb{Z}, \mathbb{Z}_k$ denote the set of all positive integers, positive integers at most k, all integers, and integers modulo k, respectively. Throughout this paper, n *denotes a positive integer.*

A path P of length $k \geq 0$ is often given in the form $P: x_0, x_1, \dots, x_k$ by the vertices successively passed through, but always considered as a subgraph; P is called an x_0,x_k-path, and $x_1, \dots, x_{k-1}$ are the interior vertices of P. A path and a circuit of length k are denoted by P_k and C_k, respectively, and P_∞ denotes a two-way infinite path, i.e. a 2-regular tree. The graph S_n arises from the complete graph K_{2n+2} on $2n+2$ vertices by deletion of the edges of a 1-factor of K_{2n+2}. For graphs G and $H, G[H]$ is defined in the following way: Take copies H_x of H for all $x \in G$ such that $H_x \cap H_y = \emptyset$ for all vertices $x \neq y$ from G. Then $G[H]$ arises from $\bigcup_{x\in G} H_x$ by addition of all edges $\bigcup_{[x,y]\in E(G)} \{[u,v] : u \in H_x \text{ and } v \in H_y\}$. – Similarly, $G+H$ arises from the union of disjoint copies G' of G and H' of H by addition of the edges $\{[u,v] : u \in G' \text{ and } v \in H'\}$.

1. Preliminary results

First some connectivity concepts. A system of x,y-paths is *openly disjoint*, if the paths are different and have pairwise no interior vertices in common. For $x \in G$ and $X \subseteq V(G-x)$, an *x,X-path* is an x,y-path for a $y \in X$ with no interior vertex in X and an *x,X-fan of order n* consists of n x,X-paths which have pairwise only x in common. The maximum number of paths in a system of openly disjoint x,y-paths in G is called *local connectivity of x and y in G* and denoted by $\kappa(x,y;G)$. The *connectivity number $\kappa(G)$ of G* is $\min_{x\neq y} \kappa(x,y;G)$, and supplementary $\kappa(K_m) = m-1$ for all non-negative integers m. In general, $\kappa(G)$ may be any cardinal number, but we will be interested only in graphs with finite connectivity number. A well known *result of K. Menger* says that *for non-adjacent vertices $x \neq y$ in G, $\kappa(x,y;G)$ is the minimum of $|T|$ for all $T \subseteq V(G-\{x,y\})$ such that x and y are in different components of $G-T$* (see, for instance, section 3.3 in [2]). A *theorem of H. Whitney* ("global version of Menger's theorem" in [2]) states $\kappa(G) := \min\{|T| : T \subseteq V(G)$ *with $G-T$ unconnected* $\}$ *for all non-complete G*. Such a set $T \subseteq V(G)$ with $G-T$ unconnected is called a *separating set of G* and a separating set T of G with $|T| = \kappa(G)$ is a *smallest separating set of G*. A union $\bigcup_{C\in\mathcal{C}'} C$ for a non-empty, proper $\mathcal{C}' \subseteq \mathcal{C}(G-T)$ where T is a smallest separating set of G is called a *fragment of G*. More exactly, it is called a *T-fragment*, and it is a *fragment of G at $t \in G$*, if it is a T-fragment for a T containing t. If $F := \bigcup_{C\in\mathcal{C}'} C$ is a T-fragment, then also $\overline{F} := \bigcup_{C\in\mathcal{C}(G-T)-\mathcal{C}'} C$ is a T-fragment. If a graph G has a finite fragment, a fragment of least vertex number is called an *atom of G*. We need some properties of fragments which have been proved in different papers, but mostly only for finite graphs. It is no problem to see that their proofs work also for infinite graphs with finite connectivity number.

Lemma 1.1 (Lemma 1 in [13] and Theorem 1 in [10] and [15]). *Let G be a graph of finite connectivity number n.*

(a) If C is an S-fragment and D is a T-fragment of G with $C \cap D \neq \emptyset$, then $|S \cap D| \geq |T \cap \overline{C}|$ holds. If, in addition, $\overline{C} \cap \overline{D} \neq \emptyset$, then $C \cap D$ is an R-fragment of G for the smallest separating set $R := (S \cap D) \cup (S \cap T) \cup (T \cap C)$.

(b) If A is an atom of G and if there is a smallest separating set T of G with $T \cap A \neq \emptyset$, then $V(A) \subseteq T$ and $|A| \leq \frac{1}{2}|T - N_G(A)| = \frac{n - |T \cap N_G(A)|}{2}$ hold.

A graph G is *n-connected*, if $\kappa(G) \geq n$, and it is called *minimally n-connected or n-minimal*, if G is n-connected, but $G - e$ is not n-connected for all $e \in E(G)$. R. Halin proved in [3] that every finite n-minimal graph has a vertex of degree n. This is immediately implied by the following property of a circuit in an n-connected graph.

Theorem 1.2 (Satz 1 in [12]). *If C is a circuit in the n-connected graph G, then there is an $e \in E(C)$ with $\kappa(G-e) \geq n$ or an $x \in V(C)$ with $d_G(x) = n$.*

Corollary 1.3 ([12]). *In every n-minimal graph $G, G - V_n(G)$ is a forest.*

For the next result we need a generalization of (n,k)-criticality. A graph G is *W-locally (n,k)-critical* for a $W \subseteq V(G)$ and positive integers n,k, if $W \cap F \neq \emptyset$ for all fragments F of G and $\kappa(G - W') = n - |W'|$ for every $W' \subseteq W$ with $|W'| \leq k$ hold. So for $W = V(G)$, we get back the concept of an (n,k)-graph. The following result was proved by T. Jordán in [5] only for W-locally (n,k)-critical, finite graphs, but the proof remains true for W finite.

Theorem 1.4 (Corollary 4 in [5]). *If G is a W-locally (n,k)-critical, non-complete graph with $2k \geq n$ and W finite, then G has $2k+2$ pairwise disjoint fragments.*

Theorem 1.4 is not true for infinite W, in general, as the $(n,1)$-graph $P_\infty[K_n]$ for $n = 1, 2$ shows. But it can be proved in the same way as Corollary 1 in [13] (conf. also Lemma 3.9) that there are no W-locally $(n,2)$-critical graphs with W infinite.

2. n-connected graphs of large order

In this section we deal only with finite graphs, unless otherwise stated explicitly. We will prove that every n-connected graph G of sufficiently large order contains a vertex set B of prescribed size so that $G - B$ is $(n-2)$-connected.

The first lemma generalizes Theorem 2 of [11] and has more or less the same proof.

Lemma 2.1. *If G is a minimally n-connected, finite graph with $n \geq 2$ and $W \subseteq V(G)$, then*

$$|W \cap V_n(G)| \geq \frac{n-1}{2n-1}\left(|W| - \frac{n|N_G(W)|-2}{n-1}\right)$$

holds.

Proof. Let $e' := |\{[x,y] \in E(G(W \cup N(W))) : d_G(x) > n \text{ and } d_G(y) > n\}|$ and $e'' := |\{[x,y] \in E(G(W \cup N(W))) : d_G(x) > n \text{ and } d_G(y) = n\}|$. Furthermore, set $b := |N_G(W)|, b_n := |V_n(G) \cap N(W)|, w := |W|$, and $w_n := |V_n(G) \cap W|$. Since by Corollary 1.3 $e' \leq w - w_n + b - b_n - 1$ and, obviously, $e'' \leq n(w_n + b_n)$, we get

(α) $2e' + e'' \leq 2w + (n-2)w_n + 2(b - b_n) + nb_n - 2 \leq 2w + (n-2)w_n + nb - 2.$

On the other side, obviously,

$$(\beta) \qquad 2e' + e'' \geq (n+1)(w - w_n)$$

holds. But (α) and (β) imply $(2n-1)w_n \geq (n-1)w - (nb-2)$. ∎

Admitting only paths of bounded interior degree, we define a distance dependent on a real number m. For vertices x, y of a finite or infinite graph G and a real m or $m = \infty$, we define $d_G^{(m)}(x,y) := \min\{||P|| : P\ x,y\text{-path in } G \text{ with } d_G(z) \leq m \text{ for all interior vertices } z \text{ of } P\}$, if there is such a path, and $d_G^{(m)}(x,y) := \infty$, if there is none. Then $d_G^{(m)}(x,y)$ is symmetric and $d_G^{(m)}(x,y) = 0$, if and only if $x = y$, but the triangle inequality is not true, in general. For $k \leq m$, we have always $d_G^{(k)}(x,y) \geq d_G^{(m)}(x,y)$ and for $m = \infty$, we get the usual metric $d_G(x,y) := d_G^{\infty}(x,y)$. For a real number r and $x \in G$, we define $B_r^{(m)}(x) := \{y \in G : d_G^{(m)}(x,y) \leq r\}$. So $B_0^{(m)}(x) = \{x\}$ and $B_1^{(m)}(x) = \overline{N}_G(x)$ for all m, and for $k \leq m, B_r^{(k)}(x) \subseteq B_r^{(m)}(x)$ holds for $x \in G$ and every integer r. Of course, these concepts depend only on $\lfloor m \rfloor$ and $\lfloor r \rfloor$, but it seems convenient in the following to admit any reals. The following upper bound for $|B_r^{(m)}(x)|$ is proved as usual.

Lemma 2.2. *For all reals $r \geq 1$ and $m > 2, |B_r^{(m)}(x)| \leq 1 + d_G(x)\frac{(m-1)^r - 1}{m-2}$ holds for all vertices x of a finite or infinite graph G.*

Proof. Obviously, $|B_r^{(m)}(x)| = |B_{\lfloor r \rfloor}^{(\lfloor m \rfloor)}(x)| \leq 1 + d_G(x) + d_G(x)(\lfloor m \rfloor - 1) + \ldots + d_G(x)(\lfloor m \rfloor - 1)^{\lfloor r \rfloor - 1} \leq 1 + d_G(x)(1 + (m-1) + \ldots + (m-1)^{\lfloor r \rfloor - 1}) \leq 1 + d_G(x)\frac{(m-1)^r - 1}{m-2}$. ∎

The next lemma provides the induction step in our proof.

Lemma 2.3. *Let G be an n-minimal, finite graph and let $B \subseteq V_n(G)$ with $\kappa(G-B)=n-2$. If F is a fragment of $G-B$ with $|F| > \frac{2n-1}{n-1} n|B|\frac{(m-1)^r-1}{m-2} + \frac{n(|B|+n-2)-2}{n-1}$ for some reals $r \geq 1$ and $m>2$, then there is a $z \in F \cap V_n(G)$ with $d_G^{(m)}(z,b)>r$ for all $b \in B$.*

Proof. By Lemma 2.2 we have $|B_r^{(m)}(b)| \leq 1+n\frac{(m-1)^r-1}{m-2}$ for all $b\in B$, hence $|F \cap \bigcup_{b\in B} B_r^{(m)}(b)| \leq |B|n\dfrac{(m-1)^r-1}{m-2}$. Since $\kappa(G-B)=n-2, n\geq 2$ follows. Then Lemma 2.1 and the assumption on F imply $|F\cap V_n(G)| \geq \frac{n-1}{2n-1}(|F| - \frac{n(|B|+n-2)-2}{n-1}) > n|B|\frac{(m-1)^r-1}{m-2}$. Hence there is a $z \in (F \cap V_n(G)) - \bigcup_{b\in B} B_r^{(m)}(b)$. ∎

Remark 2.4. The inequality for F in Lemma 2.3 is satisfied, if $|F| \geq 3m(m-1)^r, m > |B| + \frac{3}{2}(n-2)$, and $n \geq 3$, as easily checked.

Using Lemma 2.1 for $W=V(G)$ and Lemma 2.2, the proof of the following lemma is obvious and left to the reader.

Lemma 2.5. *If $v_1,\dots,v_k$ are vertices of degree n in an n-minimal, finite graph G with $|G| > \frac{2n-1}{n-1}(k+kn\frac{(m-1)^r-1}{m-2}) - \frac{2}{n-1}$ for integers $n\geq 2, k\geq 0$ and reals $m>2$ and $r\geq 1$, then there is a $v\in V_n(G)$ with $d_G^{(m)}(v,v_\kappa)>r$ for all $\kappa \in \mathbb{N}_k$. In particular, every n-minimal graph G with $|G| > \frac{2n-1}{n-1}(k+kn\frac{(m-1)^r-1}{m-2})$ contains a $W\subseteq V_n(G)$ with $|W|=k+1$ such that $d_G^{(m)}(w,w')>r$ for all $w\neq w'$ from W.*

Let $f_n(k,m,r) := \frac{2n-1}{n-1} nk\frac{(m-1)^r-1}{m-2} + \frac{n(k+n-2)-2}{n-1}$ for $m>2, n\geq 2$, and $k,r \geq 1$, denote the function occurring in Lemma 2.3. Obviously, $f_n(k,m,r) > 2nk\max\{\lfloor r \rfloor, m\} \quad > \max\{r,m\}$ for $r\geq 2$. We prove now our main result.

Theorem 2.6. *For every positive integers $n\geq 2$ and k, there is an integer $h_n(k)$ such that every n-minimal, finite graph G with $|G| \geq h_n(k)$ contains an independent $W\subseteq V_n(G)$ with $|W|=k+1$ so that $\kappa(G-W)\geq n-2$ holds.*

Proof. Since, by Lemma 2.1 (or Theorem 2 in [11]), $V_n(G)$ becomes arbitrarily large for n-minimal graphs G with $n\geq 2$, if $|G|$ is large enough, we may assume $n\geq 3$ and $k\geq 2$. We define now reals a_k and a_i, m_i, r_i for $i=k–1,\dots,1$, which guarantee that having chosen a specified set $W' \subseteq V_n(G)$ with $|W'|=i$, an atom of $G-W'$ is still larger than a_i (if $\kappa(G-W')=n-2$). We set $a_k := \frac{2n-1}{n-1} n\,k + \frac{n(n+k-2)-2}{n-1} > 4n+3$ and recursively for $i=k-1,\dots,1$, we define $m_i := a_{i+1}+n+i-2, r_i := a_{i+1}+1$ and $a_i := f_n(i,m_i,r_i)$. Since

$f_n(i,m,r) > \max\{m,r\}$ for $m>2$ and $r \geq 2$, we conclude $a_i > m_i > r_i > a_{i+1}$, hence $a_j > a_i > a_k, m_j > m_i$, and $r_j > r_i$ for all $j < i < k$. We prove now that every $h_n(k) > \frac{2n-1}{n-1}k + k\,a_1$ has the claimed property.

Let G be an n-minimal graph with $|G| > \frac{2n-1}{n-1}k + k\,a_1$. Since there is a $W' \subseteq V_n(G)$ with $|W'| = k+1$ and $d_G^{(m_1)}(w,w') > r_1$ for all $w \neq w'$ from W' by Lemma 2.5, we can find vertices $w_0, w_1, \ldots, w_{i_0}$ of degree n for an $i_0 \geq 1$ with $d_G^{(m_1)}(w_j,w_l) > r_1$ for $0 \leq j < l \leq i_0$ and $\kappa(G-\{w_0,w_1,\ldots,w_{i_0-1}\}) \geq n-1$, but $\kappa(G-\{w_0,w_1,\ldots,w_{i_0}\}) = n-2$ or $i_0 \geq k$. Since $r_1 \geq 1$, no w_j and w_l are adjacent in G and we would be done in case $i_0 \geq k$. So we may assume $i_0 < k$. Since $m_{i_0} \leq m_1$ and $r_{i_0} \leq r_1$, also $d_G^{(m_{i_0})}(w_i,w_j) > r_{i_0}$ holds for all $0 \leq i < j \leq i_0$. We enlarge now the set $\{w_0,\ldots,w_{i_0}\}$ to an independent set of $k+1$ vertices W with $\kappa(G-W) \geq n-2$ by induction. So we may assume that we have found for an i with $i_0 \leq i < k$ vertices $w_0,w_1,\ldots,w_i$ of degree n with $d_G^{(m_i)}(w_j,w_l) > r_i$ for all $0 \leq j < l \leq i$ and $\kappa(G-\{w_0,\ldots,w_i\}) = n-2$. Let us consider an atom A of $G' := G-\{w_0,\ldots,w_i\}$. Since $\kappa(G) = n$, but $\kappa(G') = n-2$, there are at least two vertices $w_j, w_l \in N_G(A) \cap \{w_o,\ldots,w_i\}$. Let us suppose $|A| \leq a_{i+1}$. Then $d_G(x) \leq |A|-1+n-2+i+1 \leq a_{i+1}+n+i-2 = m_i$ for all $x \in A$, hence $d_G^{(m_i)}(w_j,w_l) \leq |A|+1 \leq r_i$, a contradiction to the assumption on $w_o,w_1,\ldots,w_i$. So we get $|A| > a_{i+1}$. If $i+1 = k$, there is a vertex $w_k \in (A - \bigcup_{j=0}^{k-1} N_G(w_j)) \cap V_n(G)$, by definition of a_k and by Lemma 2.3 for $r=1$ and any $m>2$. If $i+1<k$, then $|A| > f_n(i+1,m_{i+1},r_{i+1})$ and by Lemma 2.3 there is a $w_{i+1} \in V_n(G) \cap A$ with $d_G^{(m_{i+1})}(w_{i+1},w_j) > r_{i+1}$ for all $0 \leq j \leq i$. Since A is an atom of G' with $|A| > a_{i+1} > \frac{n-2}{2}, \kappa(G'-x) = n-2$ for all $x \in A$ by Lemma 1.1(b). Since $r_{i+1} \geq 1$ and $r_{i+1} < r_i$ and $m_{i+1} < m_i$, we have found an independent set $W'' := \{w_0,w_1,\ldots,w_i,w_{i+1}\} \subseteq V_n(G)$ with $d_G^{(m_{i+1})}(w_j,w_l) > r_{i+1}$ for all $0 \leq j < l \leq i+1$ and $\kappa(G-W'') = n-2$. This completes the proof by induction. ∎

Remark 2.7. The statement of Theorem 2.6 with $\kappa(G-W) = n-2$ instead of $\kappa(G-W) \geq n-2$ is not true, as the complete bipartite graphs $K_{n,n+m}$ show. Theorem 2.6 is not true for $n=1$, since a 1-minimal graph, i.e. a tree, does not necessarily have more than two vertices of degree 1.

Since every n-connected, finite graph contains an n-minimal spanning subgraph, Theorem 2.6 immediately implies:

Corollary 2.8. *For all positive integers n and k, there is an integer $h_n(k)$ such that every n-connected, finite graph G with $|G| \geq h_n(k)$ contains a $W \subseteq V(G)$ with $|W| = k+1$ so that $\kappa(G-W) \geq n-2$ holds.*

As mentioned in the introduction, for $n=3$, a stronger result than Corollary 2.8 was proved in [8] with $n-1$ instead of $n-2$, and of course, this is also true for $n \le 2$. But the next example shows that for all $n \ge 4$ and all $k \in \mathbb{N}, n-2$ is best possible.

Example 2.9. First let n be even, say, $n=2p$ with $p \ge 2$ and let $m \in \mathbb{N}$ with $m > n$. Then the graph $H_m(n) := (\mathbb{Z}_m, \{[i, i+j] : i \in \mathbb{Z}_m \text{ and } j = 1, \dots, p\})$ is n-regular and $(n,2)$-critical (see [13]). Consider $W \subseteq \mathbb{Z}_m$ with $|W| \ge 2$ and set $R := H_m(n) - W$. If $\delta(R) \ge n-1$, then for every $i \in W$, $\{i+j : j = \pm 1, \dots, \pm p\} \cap W = \emptyset$, since $p \ge 2$. But this implies easily $\kappa(R) \le n-2$.

For odd $n \ge 5$, the graphs $H_m(n) := K_1 + H_m(n-1)$ are n-minimal $(n,2)$-graphs with $\kappa(H_m(n) - W) \le n-2$ for all $W \subseteq V(H_m(n))$ with $|W| \ge 2$. ∎

The function h_n constructed in the proof of Theorem 2.6 grows rapidly, but I am in doubt, if this is really necessary. Perhaps $h_n(k)$ can be taken even linear in k.

3. k-con-critical graphs

In this section we will consider the question, what of Theorem 2.6 remains true, if we search for a connected subgraph W instead of an independent set W . This suggests to consider the following concept.

Definition 3.1. A graph G is called k-con-critically n-connected, $(n,k)_c$-critical or $(n,k)_c$-graph for a non-negative integer k, if $\kappa(G - V(W)) = n - |W|$ for every connected subgraph $W \subseteq G$ with $|W| \le k$. A k-con-critical graph is an $(n,k)_c$-graph for some n.

For an $(n,k)_c$-graph G we get, in particular, $\kappa(G) = n$ for W empty. Of course, every (n,k)-graph is also an $(n,k)_c$-graph and a k-con-critical graph is k'-con-critical for all non-negative integers $k' \le k$. For $k=2$, we get back the concept of a contraction-critical n-connected graph, i.e. a graph of connectivity number n such that contracting any edge the connectivity number decreases. Since every n-regular, n-connected graph, where every edge is in a triangle, is contraction-critical, it is easy to see (and well known) that for all $n \ge 4$, there are $(n,2)_c$-graphs which are not $(n,2)$-graphs. (For $n=4$, one can use Theorems 7 and 8 from [14].) It was proved in [18] that K_{n+1} is the only finite (n,k)-graph with $k > \frac{n}{2}$. An easier proof of this result was given in [5] using Theorem 1.4. One can use this result in the same way to show the following corollary of Theorem 1.4.

Corollary 3.2. *There is no finite non-complete $(n,k)_c$-graph with $k > \frac{n}{2}$.*

Proof. We may assume $k \geq 2$ and that such a graph G is n-minimal. Let us consider a vertex z with $d_G(z) = n$ (existence by [3] or 2.1). Then, obviously, for $W := N_G(z), G - z$ would be a non-complete W-locally $(n-1, k-1)$-critical graph with $2(k-1) \geq n-1$, since every fragment F of $G - z$ has $F \cap N(z) \neq \emptyset$. This contradicts Theorem 1.4, since $|W| = |N(z)| = n < 2k$. ∎

(Notice that for $n = 3$, this is the well known consequence of Tutte's construction of 3-connected graphs [21] that every finite, 3-connected graph G with $|G| \geq 5$ has a "contractible edge". See also [19].)

In the preceding paragraphs, we have not found essentially different features of the two concepts (n,k)-critical and $(n,k)_c$-critical. But in the context we are interested in, there are some. It was proved in [13] that for every n there are only finitely many $(n,3)$-graphs. In the following, we will construct for n large enough infinitely many $(n,3)_c$-graphs. This shows that in Theorem 2.6 we cannot replace "independent" with "connected": *For every $n \geq 18$, there are n-connected, finite graphs G of arbitrarily large order which do not contain a connected subgraph W with $|W| = 3$ and $\kappa(G - V(W)) \geq n-2$.*

For this construction, the following concept is convenient.

Definition 3.3. A graph G has property $\mathcal{P}_k$, if for every path $P_{k'} \subseteq G$ with $k' \leq k$, there is a vertex $x \in G$ with $V(P_{k'}) \subseteq N_G(x)$.

If G has $\mathcal{P}_k$, for every $P_{k'} \subseteq G$ with $k' \leq k$ there is a $P_k \subseteq G$ with $P_{k'} \subseteq P_k$. Of course, every n-regular, n-connected graph with $\mathcal{P}_2$ is an $(n,3)_c$-graph, and $n \geq 3$ holds. So we will try to construct such graphs with arbitrarily large finite order.

Example 3.4. (a) If G is a graph without isolated vertices, then $G[K_s]$ has property $\mathcal{P}_2$ for $s \geq 2$.

(b) If G and H are graphs without isolated vertices, then $G + H$ has property $\mathcal{P}_2$.

For graphs G, H, and a subgraph $F \subseteq G$, we define now a graph $G\lfloor H \rfloor^F$. For every $x \in G$, let H_x be a copy of H so that $V(H_x) \cap V(H_y) = \emptyset$ for $x \neq y$. Let v_x be the vertex of H_x corresponding to $v \in H$, and for $U \subseteq V(H)$ we define $U_x := \{u_x : u \in U\}$. The graph $G\lfloor H \rfloor^F$ arises from $\bigcup_{x \in G} H_x$ by addition of all edges $\{[v_x, w_y] : [x,y] \in E(G)$ *and* $[v,w] \in E(H)\}$ *and* $\{[v_x, v_y] : v \in H$ *and* $[x,y] \in E(F)\}$.

For the construction of infinitely many finite $(n,3)_c$-graphs we need the next lemmata. Please, notice that H and G may be infinite there, but n and k are positive integers.

Lemma 3.5. *If H has property $\mathcal{P}_2$, then $G\lfloor H\rfloor^F$ has also property $\mathcal{P}_2$ for all G and $F \subseteq G$.*

Proof. Let $P: u_x, v_y, w_z$ be a path of length 2 in $G' := G\lfloor H\rfloor^F$ with $u,v,w \in H$ and $x,y,z \in G$. By definition of G', we have $u,w \in \overline{N}_H(v)$. Since H has $\mathcal{P}_2$, there is a $z \in H$ with $u,v,w \in N_H(z)$. This implies $V(P) \subseteq N_{G'}(z_y)$ by definition of G'. ∎

Lemma 3.6. *Let H be an n-regular graph with $\kappa(H) > \frac{n}{2}$, let G be a k-regular, k-connected graph for an integer k with $|G| > 2k \geq 4$, and let F be a k'-factor of G for a non-negative integer k'. If $n - \kappa(H) + k' \leq k$ and $k|H| \geq (k+1)n + k' =: r$, then $G\lfloor H\rfloor^F$ is an r-regular, r-connected graph.*

Proof. It is obvious, that $G' := G\lfloor H\rfloor^F$ is regular of degree $r := (k+1)n + k'$. First, we show that non-adjacent vertices of each "layer" H_x are r-connected in G'.

(α) *For every $x \in G$ and all non-adjacent $u_x \neq v_x$ in $H_x, \kappa(u_x, v_x; G') = r$ holds.*

Proof of (α). Abbreviating $\kappa_0 := \kappa(u,v;H) \geq \kappa(H)$, there are κ_0 openly disjoint u,v-paths $P_1, \ldots, P_{\kappa_0}$ in H. Of course, we may assume $|N(u) \cap P_i| = |N(v) \cap P_i| = 1$ for all $i = 1, \ldots, \kappa_0$. Hence, the sets $U := N(u) - \bigcup_{i=1}^{\kappa_0} V(P_i)$ and $V := N(v) - \bigcup_{i=1}^{\kappa_0} V(P_i)$ have exactly $d := n - \kappa_0$ vertices and are disjoint by definition of κ_0. If we take the corresponding u_x, v_x-paths in H_x and for every $y \in N_G(x)$ the corresponding u_x, v_x-paths "through H_y", we get altogether $(k+1)\kappa_0$ openly disjoint u_x, v_x-paths in G'.

We have to find further $(k+1)d + k'$ openly disjoint u_x, v_x-paths in G' which are also openly disjoint to the paths constructed above. For this, we plan to go from u_x over U_y and possibly over an edge $[u_x, u_y]$ or a path u_x, w_x, u_y with $w \in U$ to H_y for $y \in N_G(x)$, and continue then from H_y to an H_{z_y} for a $z_y \in N_G(y) - \{x\}$, so that all these paths do not intersect each other internally.

By assumption, $|G - \overline{N}_G(x)| \geq k$. So, by Menger's Theorem, we find k disjoint edges $[y, z_y] \in E(G - x)$ for $y \in N(x)$. Since $\overline{U} := U \cup \{u\}$ has exactly $d + 1 \leq n$ vertices, it is no problem to find a set $\overline{U}(y) \subseteq V(H_{z_y})$ such that $G'(\overline{U}_y \cup \overline{U}(y)) - (E(G'(\overline{U}_y)) \cup E(G'(\overline{U}(y))))$ has a 1-factor F_y. Since $|U| = d \leq k - k'$ by assumption, there is an injective function $f: U \to N_{G-F}(x)$.Consider now any $y \in N_G(x)$. If $y \in N_F(x)$, then there is a $u_x, \overline{U}(y)$-fan of order $d+1$ over $\overline{U}_y$ and F_y. If $y \in N_{G-F}(x)$ and $y = f(w)$ for a $w \in U$, we take again a

$u_x,\overline{U}(y)$-fan of order $d+1$ over $\overline{U}_y$ and F_y, where one path begins u_x,w_x,u_y. If $y\in N_{G-F}(x)-f(U)$, we take a $u_x,\overline{U}(y)$-fan of order d over U_y and F_y. Starting from v_x in an analogous way, we can find a set $\overline{V}(y)\subseteq V(H_{z_y})$, an injective function $g:V\to N_{G-F}(x)$ satisfying the additional condition $f(U)=g(V)$, and a $v_x,\overline{V}(y)$-fan of the same order $d+1$ or d as the $u_x,\overline{U}(y)$-fan above.

Since $\kappa(H)\geq n+1-\kappa(H)\geq n+1-\kappa_0=d+1$, by a well known form of Menger's Theorem, we can close the above $u_x,\overline{U}(y)$-paths and $v_x,\overline{V}(y)$-paths by disjoint paths in H_{z_y} to get $d+1$ or d openly disjoint u_x,v_x-paths, respectively. If we take these u_x,v_x-paths for every $y\in N(x)$, we get $kd+k'+d$ openly disjoint u_x,v_x-paths altogether. Since these paths are openly disjoint to the paths constructed in the first paragraph of this proof, we get $\kappa(u_x,v_x;G')\geq(k+1)\kappa_0+(k+1)d+k'=(k+1)n+k'$. ∎

Let us suppose that G' is not r-connected. Then (by the global version of Menger's Theorem) there is a $T\subseteq V(G')$ separating G' with $|T|=\kappa(G')<r$. We deduce some properties.

(β) *If there are $C_1\neq C_2$ in $\mathcal{C}(G'-T)$ with $C_i\cap H_{x_i}\neq\emptyset$ for an edge $[x_1,x_2]\in E(G)$, then $|T\cap H_{x_i}|\geq n$ for $i=1,2$ and $|T\cap(H_{x_1}\cup H_{x_2})|\geq|H|$. If $[x_1,x_2]\in E(F)$, then even $|T\cap H_{x_i}|\geq n+1$ holds for $i=1,2$.*

Proof of (β). Since $C_1\cap H_{x_2}=\emptyset$ by (α) and $|N_{G'}(c)\cap H_{x_2}|\geq n$ for $c\in C_1\cap H_{x_1}\neq\emptyset$, we see $|T\cap H_{x_2}|\geq n$. If $[x_1,x_2]\in E(F)$, then $|N_{G'}(c)\cap H_{x_2}|\geq n+1$, hence even $|T\cap H_{x_2}|\geq n+1$.

Consider a $z_{x_1}\in C_1\cap H_{x_1}$ which has a neighbour in $C_1\cap H_{x_1}$, say, w_{x_1}. This implies $z_{x_2}\in N_{G'}(w_{x_1})$, hence $z_{x_2}\in T$ by (α). Let $C_0:=\{z_{x_1}\in C_1\cap H_{x_1}: z_{x_1}$ isolated in $C_1\cap H_{x_1}\}$. Then $N:=N_{H_{x_1}}(C_0)\subseteq T$, hence $N_{x_2}:=\{z_{x_2}:z_{x_1}\in N\}\subseteq T$. Since H is n-regular, we have $n|C_0|\leq n|N|$, hence $|N_{x_2}|=|N|\geq|C_0|$. Since $V(H_{x_1})-V(C_1)\subseteq T$ by (α) and since $N_{x_2}\cap\{z_{x_2}:z_{x_1}\in C_1\cap H_{x_1}\}=\emptyset$, we get $|T\cap(H_{x_1}\cup H_{x_2})|\geq|H_{x_1}-V(C_1)|+|N_{x_2}|+|V(C_1\cap H_{x_1})-C_0|\geq|H_{x_1}|=|H|$. ∎

(γ) *If $T\cap H_x\neq\emptyset$, then $|T\cap H_x|\geq n$.*

Proof of (γ). Assume $T\cap H_x\neq\emptyset$. We may assume that there is a $C\in\mathcal{C}(G'-T)$ with $C\cap H_x\neq\emptyset$, since $|H_x|>n$. Since for every $t\in T$, $N_{G'}(t)\cap C'\neq\emptyset$ for all $C'\in\mathcal{C}(G'-T)$, there must be an $[x,y]\in E(G)$ and a $C'\in\mathcal{C}(G'-T)$ different from C with $C'\cap H_y\neq\emptyset$ by (α). Now (β) implies $|T\cap H_x|\geq n$. ∎

If $T\cap H_x\neq\emptyset$ for all $x\in G$, we get easily the contradiction $|T|=\sum_{x\in G}|T\cap H_x|\geq|G|n\geq(2k+1)n\geq(k+1)n+k'=r$ by (γ) and assumptions on $|G|$ and k'. Hence, there is an H_x and a $C_x\in\mathcal{C}(G'-T)$ with $H_x\subseteq C_x$. Of course, there are a $y\in G-x$ and a $C_y\in\mathcal{C}(G'-T)-\{C_x\}$ with $C_y\cap H_y\neq\emptyset$. Since $H_x\subseteq C_x$, we have $V(H_z) \subseteq V(C_x) \cup T$ for all $z \in N(x)$. So x and y are non-adjacent in G. Since G is k-connected, there are k openly disjoint x,y-paths $P_1, \ldots, P_k$ in G. Starting from x, there is a first $y_j \in P_j$ with $H_{y_j} \cap C_x = \emptyset$ by (α) for $j = 1, \ldots, k$. Since $y_j \neq x$, the predecessor x_j of y_j on P_j exists for $j = 1, \ldots, k$. Then $H_{x_j} \cap C_x \neq \emptyset$ by definition of y_j. We distinguish 3 cases.

(i) *If $x_j = x$, then $y_j \neq y$ and $V(H_{y_j}) \subseteq T$.*

Since we have seen above $V(H_z) \subseteq V(C_x) \cup T$ for all $z \in N(x), V(H_{y_j}) \subseteq T$ and $y_j \neq y$ by definition of y_j and y.

(ii) *If $x_j \neq x$ and $y_j \neq y$, then $|T \cap (H_{x_j} \cup H_{y_j})| \geq |H|$.*

If $V(H_{y_j}) \subseteq T$, this is obvious. If $V(H_{y_j}) \not\subseteq T$, there is a $C \in \mathcal{C}(G' - T)$ with $C \cap H_{y_j} \neq \emptyset$. By choice of H_{y_j}, then $C \neq C_x$, and we get $|T \cap (H_{x_j} \cup H_{y_j})| \geq |H|$ by (β).

(iii) *If $y_j = y$, then $x_j \neq x$ and $|H_{x_j} \cap T| \geq n$. If, in addition, $[x_j, y_j] \in E(F)$, even $|H_{x_j} \cap T| \geq n+1$ holds.*

Since x and y are non-adjacent (see (i)), $x_j \neq x$ is obvious. The remainder follows from (β).

If (i) or (ii) occurs for an $j \in \mathbb{N}_k$, then $|T \cap \bigcup_{z \in P_j - \{x,y\}} H_z| \geq |H|$. Hence, (i) and (ii) cannot occur for all $j = 1, \ldots, k$, since, otherwise $|T| \geq \sum_{j=1}^{k} |T \cap \bigcup_{z \in P_j - \{x,y\}} H_z| \geq k|H| \geq r$ by assumption on $|H|$. Hence (iii) occurs at least once, and $|T \cap H_y| \geq n$ by (β). If the last edge of P_j belongs to F, we get $|T \cap \bigcup_{z \in P_j - \{x,y\}} H_z| \geq n+1$ by (i), (ii) or (iii). If the last edge of P_j is not in F, we get only $|T \cap \bigcup_{z \in P_j - \{x,y\}} H_z| \geq n$. Since every edge $[z,y] \in F$ is in a P_j, we conclude $|T| \geq |T \cap H_y| + \sum_{j=1}^{k} |T \cap \bigcup_{z \in P_j - \{x,y\}} H_z| \geq n + k'(n+1) + (k-k')n = r$. This contradiction proves Lemma 3.6. ∎

Now we apply the last two lemmata to construct infinitely many $(n,3)_c$-graphs for almost all n.

Proposition 3.7. *There are infinitely many finite $(n,3)_c$-graphs for $n = 12, 15, 16$ and all $n \geq 18$.*

Proof. It is easy to find infinitely many finite, k-regular, k-connected graphs for every $k \geq 2$, which have a k'-factor for every non-negative integer $k' \leq k$.

(For instance, one can take the graphs $(\{a_1,\dots,a_m,b_1,\dots,b_m\},\{[a_i,b_{i+j}]: j\in\mathbb{N}_k\})$ for every $m\geq k$, where the indices are considered modulo m.)

For $H=K_4$, the conditions of Lemma 3.6 for k and k' are satisfied, if $k'\leq k$ and $4k\geq 3(k+1)+k'$ hold, i.e. if $k\geq 3+k'$ holds. Since K_4 has $\mathcal{P}_2$, Lemmas 3.5 and 3.6 provide an infinite series of finite $(3(k+1)+k',3)_c$-graphs for all non-negative integers k,k' with $k\geq k'+3$. Therefore, the existence of infinitely many $(n,3)_c$-graphs follows for $n=12,15,16$ and all $n\geq 18$. ∎

There are also many other examples of large $(n,3)_c$-graphs for most of the values n above, using in Lemmas 3.5 and 3.6, for instance, $H=C_m+C_m$, $H=C_m[K_2]$ or $H=T[K_2]$ with a 3-regular, 3-connected graph T with $|T|\geq 6$. But I could not decide, if there is an infinite series of finite $(n,3)_c$-graphs for any other value $n\geq 6$ than the values n in Proposition 3.7. *Perhaps, $n=12$ is the least n, for which such a series exists.*

For $n\leq 5$, the only finite $(n,3)_c$-graph is K_{n+1} by Corollary 3.2. I have checked that the only 6-regular, 6-connected graphs with $\mathcal{P}_2$ are K_7 and S_3. This together with Corollary 3.2 suggest the *conjecture that S_n is also the only non-complete $(2n,n)_c$-graph for $n\geq 3$*, as conjectured for $(2n,n)$-graphs in [13] and proved for $n\leq 6$ in [6] and [9]. But, at the moment, I cannot exclude that there are infinitely many finite $(6,3)_c$-graphs or infinite $(6,3)_c$-graphs.

I have not succeeded in constructing an infinite series of $(n,4)_c$-graphs for any n. A similar construction as for $(n,3)_c$-graphs cannot work for $(n,4)_c$-graphs. If G is a connected graph with the property that for every connected subgraph $W\subseteq G$ with $|W|=4$, there is a vertex $z\in G$ of degree n with $V(W)\subseteq N_G(z)$, then G has diameter at most 2 and $|G|\leq 1+n+n^2$. Since all known $(n,3)$-graphs of relatively large order have the corresponding property, *I conjecture that for every n, there is only a finite number of $(n,4)_c$-graphs.* In the following, we will show that the number of $(n,7)_c$-graphs is bounded by a function in n. But first we turn to infinite $(n,k)_c$-graphs.

It was proved in [13] that every $(n,2)$-graph is finite. In Example VII in [15], an infinite $(2,2)_c$-graph was pointed out. In a similar way, one can construct infinite $(n,2)_c$- graphs for every $n\geq 2$.

Example 3.8. For any $n\geq 2$, let T_{n+1} be the $(n+1)$-regular tree. Let K_x be a complete graph on $n+1$ vertices for every $x\in T_{n+1}$ such that $K_x\cap K_y=\emptyset$ for $x\neq y$. For every $x\in T_{n+1}$, let $f_x: E_T(x)\to V(K_x)$ be a bijective function. Then the graph G_n arises from $\bigcup_{x\in T_{n+1}} K_x$ by identifying $K_x-\{f_x(e)\}$ in any bijective manner with $K_y-\{f_y(e)\}$ for every $e=[x,y]\in E(T_{n+1})$. It is easily seen, that G_n is an $(n,2)_c$-graph. ∎

All vertices of the graph G_n in this example have infinite degree. Therefore, it is not possible to construct an infinite $(n,3)_c$-graph in this way, as the following lemma shows.

Lemma 3.9. *Every $(n,3)_c$-graph is locally finite.*

Proof. Let G be an infinite $(n,3)_c$-graph and $x \in G$. Of course, there is a smallest separating set T' of G containing x. The union of n openly disjoint t,t'-path in G for every $t \neq t'$ from T provides a *finite* $G' \subseteq G$ containing T such that $\kappa(t,t';G') \geq n$ holds for all $t \neq t'$ from T. Let us suppose that $d_G(x)$ is infinite. Then there is a $y_1 \in N_G(x) - V(G')$. Let $C_1 \in \mathcal{C}(G-T)$ contain y_1. There are a $C_2 \neq C_1$ in $\mathcal{C}(G-T)$ and a $y_2 \in N_G(x) \cap C_2$. Since G is an $(n,3)_c$-graph, there is a smallest separating set $T' \supseteq \{y_1, x, y_2\}$ in G. Since $N_G(y_i) \subseteq V(C_i) \cup T$ for $i=1,2$, every component C' of $G-T'$ has $C' \cap T \neq \emptyset$. So we get the contradiction, that $T' \cap G'$ with $|T' \cap G'| < n$ separates two vertices of T in G'. ∎

But one can construct infinite $(n,3)_c$-graphs for all n large enough, analogous to Proposition 3.7.

Proposition 3.10. *There are infinite $(n,3)_c$-graphs for $n = 12, 15, 16$ and all $n \geq 18$.*

Proof. We apply again Lemmas 3.5 and 3.6 for $H = K_4$, but now for infinite G. It is no problem to find infinite, k-regular, k-connected graphs for every integer $k \geq 3$, which have a k'-factor for every non-negative integer $k' \leq k$. We will give an easy example.

Let $K_{k,k}$ be the complete bipartite graph on independent vertex sets $\{a_1, \dots, a_k\}$ and $\{a_{k+1}, \dots, a_{2k}\}$, and define $D := K_{k,k} - \{[a_i, a_{k+i}] : i \in \mathbb{N}_k\}$. Let D_x be disjoint copies of D for $x \in P_\infty$, where $a_i^x \in D_x$ corresponds to a_i for $i = 1, \dots, 2k$. Let us number the vertices $x_i (i \in \mathbb{Z})$ of P_∞ along P_∞ and let G_k arise from $\bigcup_{x \in P_\infty} D_x$ by addition of all edges $\{[a_j^{x_i}, a_{k+j}^{x_{i+1}}] : i \in \mathbb{Z}$ *and* $j \in \mathbb{N}_k\}$. It is easily checked that the k-regular graph G_k is k-connected for $k \geq 3$, and it has obviously a k'-factor for every non-negative integer $k' \leq k$.

The existence of infinite $(n,3)_c$-graphs follows now in the same way and for the same values n as in Proposition 3.7. ∎

Notice that the graphs G contsructed in Proposition 3.10 (in the same way as those of Proposition 3.7) have the stronger property that the connectivity number decreases by at least 3 deleting any vertex set S with $|S| >= 3$ and $G(S)$ connected.

There are again many other possibilities to construct infinite $(n,3)_c$-graphs using Lemmas 3.5 and 3.6 for most of the above values n. But we can also take the k-regular tree T_k for any $k \geq 2$ as G in Lemma 3.6, a k'-factor F of T_k for any $k' \leq k$, and an H as in Lemma 3.6, but with $|H| \geq (k+1)n+k'$. Then it is checked in a similar (but somewhat easier) way that $T_k\lfloor H \rfloor^F$ is $((k+1)n+k')$-connected. Again, I could not decide for other values $n \geq 6$ than in Proposition 3.10, if there are infinite $(n,3)_c$-graphs, but I intend to believe that $n = 12$ is the least n for which some exist.

Let us now consider the values $n < 6$. In Example 3.8, we have constructed infinite $(n,2)_c$-graphs for all $n \geq 2$, and, obviously, there are infinite $(n,1)_c$-graphs for all $n \geq 1$. But we will show in the following that Corollary 3.2 is true without the assumption "finite" for all $k \geq 3$. In particular, this implies that there are no infinite $(n,3)_c$-graphs for $n < 6$. We need some further lemmata.

Lemma 3.11. *Let z be a vertex of finite degree in the graph G of connectivity number n, and assume that for every $[z,x] \in E_G(z)$, there is a smallest separating set T of G with $\{x,z\} \subseteq T$. Then there is a fragment F at z with $|F| \leq \frac{n-1}{2}$.*

Proof. Let a fragment F_0 of $G' := G - z$ be chosen in the following way: If G' has finite fragments, let F_0 be an atom of G'. If G' has no finite fragments, choose F_0 with $|F_0 \cap N_G(z)| = \min\{|F \cap N_G(z)| : F \textit{ fragment of } G'\}$. We will prove $|F_0| \leq \frac{n-1}{2}$. Since the proof runs as usual, we will be brief.

F_0 is also a fragment of G at z, since $\kappa(G') = n-1$ by assumption on z. Hence, there are an $x \in N_G(z) \cap F_0$ and by assumption, a smallest separating set T of G with $\{x,z\} \subseteq T$. Let F denote any T-fragment of G and set $T_0 := N_G(F_0)$. If $F_0 \cap F \neq \emptyset$ and $\overline{F}_0 \cap \overline{F} \neq \emptyset$, Lemma 1.1(a) implies that $F_0 \cap F$ is a fragment of G'. But, obviously, $V(F_0 \cap F) \subseteq V(F_0 - x)$ and, hence, $|N_G(z) \cap F_0 \cap F| < |N_G(z) \cap F_0|$. In any case, this contradicts the definition of F_0. So we conclude that $V(F_0) \subseteq T$ or $V(\overline{F}_0) \subseteq T$ or there is a T-fragment F with $V(F) \subseteq T_0$. So F_0 is finite, hence an atom of G' and we get $|F_0| \leq \frac{n-1}{2}$ by Lemma 1.1(b). ∎

Lemma 3.12. *Let G be an infinite graph with $\kappa(G) = n$ which has a finite fragment. Then there is a finite $E_0 \subseteq E(G)$ with $\kappa(G - E_0) = \delta(G - E_0) = n$.*

Proof. Consider $\mathcal{E}_0 := \{E' \subseteq E(G) : E' \textit{ finite and } \kappa(G - E') = n\}$. Define $f_0 := \min\{||F'|| + |E_{G-E'}(F')| : F' \textit{ fragment of } G - E' \textit{ for } E' \in \mathcal{E}_0\}$ and choose an $E_0 \in \mathcal{E}_0$ and a fragment F_0 of $G_0 := G - E_0$ such that $||F_0|| + |E_{G_0}(F_0)| = f_0$, and set $T_0 := N_{G_0}(F_0)$. If $|F_0| = 1, E_0$ has the wanted property. So we assume $|F_0| > 1$. Then F_0 is connected, $|N_{G_0}(t) \cap F_0| \geq 2$ for all $t \in T_0$, and $\delta(G_0) > n$. Hence, there is a circuit C in $G_0(V(F_0) \cup \{t\})$ for $t \in T_0 \neq \emptyset$. Since $\delta(G_0) > n$, there is an $e\in E(C)$ with $\kappa(G_0-e)=n$ by (1-2). But then $E_0\cup\{e\}\in\mathcal{E}_0$ and $F':=F_0-e$ is a fragment of $G':=G-(E_0\cup\{e\})$ with $||F'||+|E_{G'}(F')|<f_0$, a contradiction to the definition of f_0. ∎

Proposition 3.13. *If G is an $(n,k)_c$-graph with $k\geq 3$ and $k>\frac{n}{2}$, then G is isomorphic to K_{n+1}.*

Proof. Suppose $|G|\geq n+2$. Then G is infinite by Corollary 3.2, but locally finite by Lemma 3.9. Since $k\geq 3, G$ is contraction-critical, hence, G has finite fragments by Lemma 3.11. Therefore, Lemma 3.12 provides the existence of an $E_0\subseteq E(G)$ with $\kappa(G-E_0)=n$ such that there is a $z\in G$ with $d_{G-E_0}(z)=n$. Of course, $G-E_0$ is also $(n,k)_c$-critical. Now we can continue as in Corollary 3.2 to complete the proof. ∎

I have not found an infinite $(n,4)_c$-graph for any n, and I conjecture there is none. But I could only show that there are no infinite $(n,k)_c$-graphs for $k\geq 7$.

Proposition 3.14. *Every $(n,7)_c$-graph G is finite, has diameter at most 4, and $|G|<4n^4$.*

Proof. Let $G\not\cong K_{n+1}$ be an $(n,7)_c$- graph, in particular, $n\geq 7$. Choose a path $P_\ell\subseteq G$ with $1\leq\ell\leq 5$, say, an x,y-path. Since G is locally finite by Lemma 3.9 and 7-con-critical, we can apply Lemma 3.11 to $G':=G-V(P-y)$ and y. So we find a fragment F of G' at y with $|F|\leq\frac{n-\ell-1}{2}$, since $\kappa(G')=n-\ell$. Since F is also a fragment of G, there are $x'\in N_G(x)\cap F$ and $y'\in N_G(y)\cap F$. Since $|F|\leq\frac{n-2}{2}$, we have $\overline{N}_G(x')\cap\overline{N}_G(y')\neq\emptyset$, and there is an x,y-path of length at most 4 in G. So G has diameter at most 4 and $d^{(\frac{3n-4}{2})}(x,y)\leq\frac{n}{2}$, since there is an x,y-path of length at most $|F|+1$ "through F". Therefore, G is finite by Lemma 2.2. If we choose x with $d(x)<\frac{3n}{2}$, Lemma 2.2 provides also a bound for $|G|$ by a function in n, but a very bad one. We will get a better one in the next paragraph.

We may assume G n-minimal. Then there is an $x\in G$ of degree n by Lemma 2.1. Consider any $y\in G$. Since G has diameter at most 4, there is an x,y-path P of length at most 4 in G. We will even show $d^{(\frac{3n-7}{2})}(x,y)\leq 4$.

There is a connected $W\subseteq G$ with $P\subseteq W$ and $|W|=5$. Set $G_1:=G-V(W)$ and consider an atom A_1 of G_1. Since G is $(n,7)_c$-critical, $\kappa(G_1)=n-5$ holds and A_1 is a fragment of G, say, a T_1-fragment with $T_1\supseteq V(W)$. Then there are $x_1\in N_G(x)\cap A_1$ and $y_1\in N_G(y)\cap A_1$. Since $G(W\cup y_1)$ is connected, there is a smallest separating set T' of G with $T'\supseteq V(W)\cup\{y_1\}$, and for every such set $T', V(A_1)\subseteq T'$ and $|A_1|\leq\frac{n-|T_1\cap T'|}{2}\leq\frac{n-5}{2}$ holds by Lemma 1.1(b). Then

$G_2 := G-(V(W)\cup V(A_1))$ has $\kappa(G_2)=n-|A_1|-5$. Let A_2 be an atom of G_2. Since A_2 is also a fragment of G_1 and G, we have $a_2:=|A_2|\geq|A_1|=:a_1$, since $A_1$is an atom of G_1, and $|T_2|=n$ and $T_2\supseteq V(W)\cup V(A_1)$ holds for $T_2=N_G(A_2)$. In particular, there is a $y_2\in N_G(y_1)\cap A_2$. Since $G(V(W)\cup\{y_1,y_2\})$ is connected and has 7 vertices, there is a smallest separating T of G with $T\supseteq V(W)\cup\{y_1,y_2\}$. Then Lemma 1.1(b) implies $V(A_1)\subseteq T$, since $y_1\in T$, and, furthermore, $|A_2|\leq\frac{n-5-|A_1|}{2}$, since $y_2\in T$ and A_2 is an atom of G_2. If $A_2\cap\overline{A}_1\neq\emptyset$, then $|A_2\cap T_1|\geq a_1$ by Lemma 1.1(a). Since $a_2\geq a_1$, so $|A_2\cap T_1|\geq a_1$ in any case, since $A_2\cap A_1=\emptyset$. But this implies $\overline{N}_G(x_1)\cap\overline{N}_G(y_1)\cap(A_1\cup A_2)\neq\emptyset$, since otherwise we had $2n+2\leq|\overline{N}(x_1)|+|\overline{N}(y_1)|\leq 2(n-|A_2\cap T_1|)+|A_1|+|A_2\cap T_1|\leq 2n$. Hence there is an x,y-path of length at most 4 with all interior vertices in $A_1\cup A_2$. This proves $d^{(\frac{3n-7}{2})}(x,y)\leq 4$ and Lemma 2.2 implies $|G|\leq 1+n\frac{(\frac{3n-9}{2})^4-1}{\frac{3n-11}{2}}<n(\frac{3n-7}{2})^3$. ∎

Proposition 3.14 says that every n-connected graph G of sufficiently large order contains a connected $W\subseteq G$ with $|W|=7$ such that $\kappa(G-V(W))\geq n-6$ holds. This could be a hint that the following is true.

Conjecture 3.15. There are a function $f:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ and an integer d so that for all $(n,k)\in\mathbb{N}\times\mathbb{N}$, every n-connected graph G with $|G|\geq f(n,k)$ contains a connected $W\subseteq G$ with $|W|=k$ such that $\kappa(G-V(W))\geq n-d$ holds.

Proposition 3.7 shows $d\geq 3$, if it exists. I should conjecture that d is the smallest integer k for which only finitely many $(n,k+1)_c$-graphs exist for every n.

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Added in proof. *In the meantime, I have succeeded in proving both the conjectures stated in the paragraphs before Proposition 3.14 and Example 3.8: every $(n,4)_c$-graph is finite, and for every n, there is only a finite number of $(n,4)_c$-graphs. This paper "On k-con-critically n-connected graphs" has appeared in the Journal of Combinatorial Theory (B)* **86** *(2002).*

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